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Classical and Quantum Instantons in Yang-Mills Theory in the Background of de Sitter Spacetime

Hongsu Kim and Sung Ku Kim

Department of Physics

Ewha Women's University, Seoul 120-750, KOREA

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Abstract

Instantons and their quantisation in pure Yang-Mills theory formulated in the background of de Sitter spacetime represented by spatially- closed ($k = +1$) Friedmann-Robertson-Walker metric are discussed. As for the classical treatment of the instanton physics, first, explicit instanton solutions are found and next, quantities like Pontryagin index and the semiclassical approximation to the inter-vacua tunnelling amplitude are evaluated. The Atiyah-Patodi-Singer index theorem is checked as well by constructing explicitly the normalizable fermion zero modes in this de Sitter spacetime instanton background. Finally, following the kink quantisation scheme originally proposed by Dashen, Hasslacher and Neveu, the quantisation of our instanton is performed. Of particular interest is the estimate of the lowest quantum correction to the inter-vacua tunnelling amplitude arising from the quantisation of the instanton. It turns out that the inter-vacua tunnelling amplitude gets enhanced upon quantizing the instanton.

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I. Introduction

It is well-known that topologically degenerate vacuum structure of non-abelian gauge theories opened up our eyes to the profound and new aspects of non-perturbative regime of the theories such as the physics of instantons and the mechanism of quark confinement. Particularly, the instanton physics in the pure Yang-Mills (YM) gauge theory formulated in flat spacetime has been thoroughly studied in the literature [1] at least semiclassically. In the present work, we discuss the classical and quantum instanton physics in the pure YM theory formulated in the background of de Sitter spacetime. The formulation of scalar and spinor field theory (particularly their quantum field theory) in the fixed background of de Sitter spacetime has long been a center of interest and actually much work [3] has been done associated with this topic. Therefore, it is somewhat curious that relatively little attempt [2] has been made toward the formulation of vector gauge theories particularly that of YM gauge theory in the same de Sitter background spacetime. And partly, this state of affair has been the motivation of the present work. There is, however, a remarkable feature that distingushes the formulation of YM gauge theory in de Sitter spacetime from that of scalar or spinor theory in the same de Sitter spacetime. Suppose one starts with the Einstein-Yang-Mills theory in the presence of the cosmological constant and treat both the gravity sector and the Yang-Mills matter sector on equal footing. As long as we restrict our interest to instanton solutions in this system, we need to look for solutions to (anti)self-dual equation for YM field strength. Then what happens is that for the (anti)self-dual field strength, the YM energy-momentum tensor vanishes identically [2] in the Euclidean signature. This indicates that the YM (matter) field does not disturb the spacetime geometry while the geometry still does have effect on the YM field. As a result, the geometry, which is left intact by the YM field, effectively serves as a "background" spacetime which can be chosen somewhat at our will and here in this work, we take it to be the de Sitter spacetime (since we included the cosmological constant). And in this work, the metric for this background de Sitter spacetime is chosen to be that of the spatially-closed FRW having the Lorentzian topology of $R \times S^3$ (with S^3 being the topology of the spatial section) and the Euclidean

topology of S^4. Thus it has SO(4)-symmetry and hence the dynamical YM field put on it should have the same SO(4)-symmetry as well. Then noticing that the SU(2) group manifold is also S^3 just like it is the case for the geometry of the spatial section of the manifold, one may choose a "common" basis for both the group manifold and the spatial section of the spacetime manifold. And this indicates that there will be "mixing" between the group index in the YM field and the frame index. Namely we can employ an analogue of the 'tHooft- Polyakov's "hedgehog" ansä̋tz for the monopole solutions [4] in Yang-Mills-Higgs theory. This high degree of "built-in" symmetry, then reduces the system effectively to a one-dimensional system of a self-interacting scalar field (namely, a kind of scalar ϕ^4-theory) with potential of the structure of that of "double-well" whose vacuum has two-fold degeneracy. Of course from this point on, one may proceed to carry out the quantisation of the one-dimensional scalar ϕ^4-type reduced system as a mean to formulate the quantum YM gauge theory in de Sitter background spacetime. However, since associated with this degeneracy in vacuum of the theory, of central interest is the physics of instanton, we, instead, explore the instanton physics of this system in the present work. As a matter of fact, there had been some works [2] on concrete study of YM instanton solutions in curved spacetime. Eguchi and Freund [2] considered the YM instanton in conformally-flat general spacetimes and Charap and Duff [2] discussed it in (maximally-extended) Schwarzschild spacetime. The detailed account of relationships between our present work and those of these authors will be given at the end of sect.II and sect.III respectively. In classical terms, instanton is a gauge field configuration which interpolates between two degenerate but distinct vacua. Or more rigorously, it is a classical solution to the Euclidean equation of motion that makes dominant contribution to the inter-vacua tunnelling amplitude. As a classical treatment of this instanton physics, first, explicit instanton solutions will be found. Next, quantities like Pontryagin index representing the instanton number and the semiclassical approximation to the vacuum-to-vacuum tunnelling amplitude will be evaluated. And lastly, the Atiyah-Patodi-Singer index theorem will be confirmed by constructing explicitly the normalizable fermion zero modes in this instanton background. Then follows the quantum treatment of

the instanton physics. As will be shown in the text later on, since the Euclidean time is just another "spacelike" coordinate, the Euclidean action of the reduced one-dimensional system may be viewed as the potential energy or the Hamiltonian of a system of "static", self-interacting scalar field. As a consequence, one can directly apply the standard, conventional soliton (particularly kink) quantisation formalism developed in original papers [5,6] to the quantisation of our instanton. Among various quantisation techniques, we shall employ, in the present work, that of Dashen, Hasslacher and Neveu [5]. As is well-known, in the context of this soliton quantisation scheme, the leading quantum correction corresponds to the contribution of a set of approximate harmonic oscillator states. Thus energy levels of quantized instanton wll be given. Finally, as a result of central importance in this work, we shall provide the lowest order quantum correction to the Euclidean instanton action and hence to the vacuum-to-vacuum tunnelling amplitude arising from the quantisation of the instanton. To summarize the result, the Euclidean action of the quantized instanton is lower than that of the classical instanton. And this suggests that in the context of quantized instanton, the inter-vacua tunnelling amplitude gets enhanced compared to what happens in the context of classical instanton. This paper is organized as follows : In sect.II, general formalism for the pure YM theory in de Sitter background spacetime is provided. In sect.III, we give a classical treatment of the instanton physics in this system. Sect.IV is particularly prepared for the confirmation of Atiyah-Patodi- Singer index theorem in the context of our system. Sect.V is devoted to the formal quantisation of our instanton and finally in sect.VI, we summarize the results of our study.

II. General Formalism

As mentioned in the introduction, we would like to discuss the physics of classical instanton solution in pure YM theory formulated in de Sitter background spacetime represented by the spatially-closed ($k = +1$) FRW-metric. Thus we begin with the action governing our system, namely the Einstein-Yang-Mills theory in the presence of the (positive) cosmological constant

$$S_{EYM} = \int d^4x\sqrt{g}[\frac{1}{16\pi G}R - \Lambda - \frac{1}{4g_c^2}F^a_{\mu\nu}F^{a\mu\nu}], \quad (1)$$
$$I_{EYM} = \int d^4x\sqrt{g}[\Lambda - \frac{1}{16\pi G}R + \frac{1}{4g_c^2}F^a_{\mu\nu}F^{a\mu\nu}]$$

in Lorentzian and Euclidean signature respectively. The classical field equations which result from extremizing the EYM theory action above is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 8\pi G\Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$
$$T_{\mu\nu} = \frac{1}{g_c^2}[F^a_{\mu\alpha}F^{a\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}(F^a_{\alpha\beta}F^{a\alpha\beta})],$$
$$D_\mu[\sqrt{g}F^{a\mu\nu}] = 0 \quad (2)$$

where we employed the convention $A_\mu = A^a_\mu(-iT^a)$ and $F_{\mu\nu} = F^a_{\mu\nu}(-iT^a)$ (with $T^a = \tau^a/2$, τ^a being Pauli spin matrices obeying the SU(2) Lie algebra $[T^a, T^b] = i\epsilon^{abc}T^c$ and the normalization $Tr(T^aT^b) = \delta^{ab}/2$) in which the YM field strength and gauge-covariant derivative are given respectively by $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu$, $D^{ac}_\mu = \partial_\mu\delta^{ac} + \epsilon^{abc}A^b_\mu$. $a, b, c = 1, 2, 3$ denote SU(2) group indices and g_c is the YM gauge coupling constant.

As mentioned in the introduction, we are particularly interested in the solution to (anti)self-dual equation in Euclidean signature to find the de Sitter spacetime version of the YM instanton. Then the Euclidean YM field energy-momentum tensor vanishes identically, $T_{\mu\nu} = 0$ and the Einstein field equation reduces to that of de Sitter spacetime, $R_{\mu\nu} - g_{\mu\nu}R/2 + 8\pi G\Lambda g_{\mu\nu} = 0$. In order to represent this de Sitter spacetime, now we employ the spatially-closed ($k = +1$) FRW-metric given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{AB}e^A\otimes e^B \quad (3)$$
$$= [-N^2(t)dt^2 + a^2(t)\sigma^a\otimes\sigma^a]$$
$$= [N^2(\tau)d\tau^2 + a^2(\tau)\sigma^a\otimes\sigma^a]$$

in Lorentzian and Euclidean signature respectively and where $N(t)$ and $a(t)$ are lapse function and scale factor respectively. First in the gravity sector (which is left intact by the YM field), there is a gauge arbitrariness which amounts to the invariance of the curvature under the 4-dim. diffeomorphisms, i.e., general coordinate transformations. And this 4-dim.

diffeomorphism consists of the time-reparametrization corresponding to possible different choices for the lapse function $N(t)$ and the 3-dim. diffeomorphism corresponding to the freedom in choosing coordinates for the left-invariant basis 1-forms $\{\sigma^a\}$ representing the metric on the spacelike hypersurface S^3. Here in this work, our choice for the gauge-fixing will be determined as follows : since this background spacetime metric has the Lorentzian topology of $R \times S^3$ and the Euclidean topology of S^4, it has SO(4)-symmetry. Thus in order to take advantage of this high degree of symmetry, we shall employ the Euler angle coordinates (θ, ϕ, ψ) parametrizing the spatial section of the manifold, S^3. This is the gauge fixing associated with the 3-dim. diffeomorphism. Next, concerning the time-reparametrization freedom, we shall mainly employ two alternative gauges $N(t) = 1$ and $N(t) = a(t)$, i.e., the so-called "conformal-time" gauge. In the first gauge $N(t) = 1$, the scale factor satisfying the Einstein equation for de Sitter spacetime is $a(t) = \frac{1}{\kappa}\cosh(\kappa t)$ ($a(\tau) = \frac{1}{\kappa}\cos(\kappa\tau)$ in Euclidean time $\tau = it$) where $\kappa = \sqrt{8\pi G\Lambda/3}$ while in the second gauge $N(t) = a(t)$, the corresponding scale factor is given by $a(t) = 1/\kappa \cos t$ ($a(\tau) = 1/\kappa \cosh\tau$ in Euclidean signature). Next, for reasons that will become clear later on, throughout this work, we shall mainly work with "non-coordinate" basis with indices $A, B = 0, a$ ($a = 1, 2, 3$) rather than with coordinate basis with indices $\mu, \nu = t, \theta, \phi, \psi$ where (θ, ϕ, ψ) are again the Euler angles. The non-coordinate basis 1-forms can be read off from the metric given in eq.(3) as

$$e^A = \{e^0 = Ndt, \quad e^a = a\sigma^a\} \tag{4}$$

where $\{\sigma^a\}$ ($a = 1, 2, 3$) form a basis on the 3-sphere S^3, as mentioned, satisfying the SU(2) "Maurer-Cartan" structure equation

$$d\sigma^a + \epsilon^{abc}\sigma^b \wedge \sigma^c = 0. \tag{5}$$

In our gauge-fixing, σ^a's are represented in terms of 3-Euler angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$, parametrizing S^3 as

$$\sigma^1 = -\frac{1}{2}(sin\psi d\theta - cos\psi sin\theta d\phi),$$

$$\sigma^2 = \frac{1}{2}(cos\psi d\theta + sin\psi sin\theta d\phi), \tag{6}$$
$$\sigma^3 = -\frac{1}{2}(d\psi + cos\theta d\phi).$$

For later use, we also write down the associated vierbein and its inverse using the definition, $e^A = e^A_\mu dx^\mu$, $e^A_\mu e^\mu_B = \delta^A_B$ and $e^\mu_A e^A_\nu = \delta^\mu_\nu$ where $x^\mu = (\tau, \theta, \phi, \psi)$

$$e^A_\mu = \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & -\frac{a}{2}sin\psi & \frac{a}{2}cos\psi sin\theta & 0 \\ 0 & \frac{a}{2}cos\psi & \frac{a}{2}sin\psi sin\theta & 0 \\ 0 & 0 & -\frac{a}{2}cos\theta & -\frac{a}{2} \end{pmatrix} \quad , \quad e^\mu_A = \begin{pmatrix} \frac{1}{N} & 0 & 0 & 0 \\ 0 & -\frac{2}{a}sin\psi & \frac{2}{a}cos\psi & 0 \\ 0 & \frac{2}{a}\frac{cos\psi}{sin\theta} & \frac{2}{a}\frac{sin\psi}{sin\theta} & 0 \\ 0 & -\frac{2}{a}\frac{cos\psi cos\theta}{sin\theta} & -\frac{2}{a}\frac{sin\psi cos\theta}{sin\theta} & -\frac{2}{a} \end{pmatrix}. \tag{7}$$

Thus far we have discussed the choice of ansätz for the metric (i.e., $k = +1$ FRW-metric which is SO(4)-symmetric) and the gauge-fixing for gravity sector. Next, we turn to the choice of ansätz for the YM gauge potential and the SU(2) gauge-fixing. And here, our general guideline is that since the background de Sitter spacetime metric is chosen to possess SO(4)-symmetry, the dynamical YM field put on it should have the SO(4)-symmetry as well. Then next, note that the SU(2) group manifold is also S^3 just as it is the case for the geometry of the spatial section of the spacetime manifold. Thus one may choose the left-invariant 1-form $\{\sigma^a\}$ as the "common" basis for both the group manifold and the spatial section of the spacetime manifold. And this indicates that there is now "mixing" between the group index in the YM field and the non-coordinate frame basis index since we, as mentioned earlier, choose to work with non-coordinate basis. Now an appropriate choice of YM gauge potential ansätz incorporating all of these conditions is [7]

$$A^a = A^a_\mu dx^\mu = [1 + H(t)]\sigma^a. \tag{8}$$

Of course in taking this YM gauge potential ansätz, we implicitly chose the "temporal gauge-fixing" $A_t = 0$ (or $A_0 = 0$ in non-coordinate basis). This gauge choice is indeed natural since the background spacetime metric is homogeneous and isotropic thus depends only on time coordinates, there is no gauge freedom associated with the space-dependent gauge transformation. By now, it should be clear that it is more appropriate to work with

non-coordinate basis. And in the formulation employing the use of non-coordinate basis, various equations involved should be put in differential forms. To be more specific, the definition for the YM field strength takes the form, $F^a = dA^a + \frac{1}{2}\epsilon^{abc}A^b \wedge A^c$ which, using the gauge potential ansätz above, is computed to be

$$F^a_{0b} = \frac{\dot{H}}{Na}\delta^a_b, \tag{9}$$
$$F^a_{bc} = \frac{(H^2-1)}{a^2}\epsilon^{abc}$$

where "dot" denotes the derivative with respect to Lorentzian time t. Next, the classical YM field equation and the Bianchi identity are

$$D\tilde{F} = d\tilde{F} + A \wedge \tilde{F} - \tilde{F} \wedge A, \tag{10}$$
$$DF = dF + A \wedge F - F \wedge A$$

respectively with "tilde" denoting the Hodge dual. As far as the classical treatment of the system is concerned, one is mainly interested in solving the classical YM field equation. Thus particularly associated with the instanton physics that we shall discuss later on, it seems worth noticing that in the pure YM theory, the solutions to (anti) self-dual equation

$$F^a = \pm\tilde{F}^a, \tag{11}$$

which is just 1st order differential equation, are automatically solutions of the classical YM field equation (owing to the Bianchi identity) as well as the minima of the Euclidean YM theory action. Another point to mention is that the SO(4)-symmetric ansätz for the YM gauge potential chosen above does obey the Bianchi identity as it should. For later use, we write down the (anti) self-dual equation above in terms of the SO(4)-symmetric ansätz for the metric and the YM gauge potential

$$\frac{H'}{Na} = \pm\frac{(H^2-1)}{a^2} \tag{12}$$

where now "prime" means the derivative with respect to the Euclidean time τ. In order to have a qualitative insight into our system prior to all quantitative calculations, we first

rewrite the action of our system given earlier in terms of the SO(4)-symmetric ansătz for the background spacetime metric and the YM gauge field. Thus using

$$(F^a_{\mu\nu})^2 = (\eta^{AC}\eta^{BD}F^a_{AB}F^a_{CD}) = 6[-(\frac{\dot{H}}{Na})^2 + (\frac{H^2-1}{a^2})^2] \tag{13}$$

it follows

$$\begin{aligned} S_{YM} &= \frac{r_0^2}{2}\int dt[(\frac{a}{N})\dot{H}^2 - (\frac{N}{a})(H^2-1)^2] \\ &= \frac{r_0^2}{2}\int d\tilde{t}[(\frac{dH}{d\tilde{t}})^2 - (H^2-1)^2] \end{aligned} \tag{14}$$

where we introduced $d\tilde{t} = (N/a)dt$ and defined $r_0^2 = 6\pi^2/g_c^2$. Note that this pure YM system put in the background of de Sitter space represented by the spatially-closed ($k = +1$) FRW metric has been reduced to a one-dimensional system of a particle of unit mass with the potential given by

$$\tilde{V}(H) = \frac{1}{2}\tilde{U}(H) = \frac{r_0^2}{2}(H^2-1)^2 \tag{15}$$

which has the "double-well" structure. Here and henceforth we introduce the notations for the "potential", $U(H) \equiv (H^2-1)^2 = 2V(H)$ and $\tilde{U}(H) = r_0^2 U(H) = 2\tilde{V}(H)$. Since the minimum of the potential, i.e., the vacuum has two-fold degeneracy at $H = \pm 1$, readily we anticipate possible quantum tunnelling phenomenon between the two degenerate vacua. Thus in order to study this vacuum-to-vacuum tunnelling, we reformulate this system in Euclidean time obtained by the Wick rotation $\tilde{\tau} = i\tilde{t} = i\int dt(\frac{N}{a}) = \int d\tau(\frac{N}{a})$. Then the Euclidean action is given by

$$I_{YM} = -iS_{YM} = \frac{r_0^2}{2}\int d\tilde{\tau}[H'^2 + U(H)] \tag{16}$$

where again the prime denotes the derivative with respect to the Euclidean time $\tilde{\tau}$ while the overdot we used earlier denotes that with respect to the Lorentzian time t. Upon extremizing this action with respect to the field H representing the YM gauge field we now get the Euclidean equation of motion

$$H'' = \frac{1}{2}\frac{\partial U}{\partial H}. \tag{17}$$

It is well-known that even without explicitly solving this equation of motion, one can easily determine the qualitative features of the solution which will turn out to be the instanton solution. Thus to do so, we consider the "first integral" of the Euclidean equation of motion given above

$$\frac{1}{2}H'^2 - \frac{1}{2}U(H) = E \tag{18}$$

where E is the integration constant. This first integral equation describes a system of unit mass particle with total energy E moving in the "inverted" potential $-V(H) = -\frac{1}{2}U(H)$. Obviously, the motion of particle with zero total energy, $E = 0$, which is of our interest, will be that the particle starts (say, at $\tilde{\tau} = -\infty$) on top of one hill and moves to the top of the other (at $\tilde{\tau} = +\infty$). Since this behavior of the particle (with position H) in this mechanical problem corresponds to the behavior of the solution $H(\tau)$ of the Euclidean equation of motion, we can expect that there will be a solution of instanton type. For example, the solution of the equation of motion satisfying the boundary condition $\lim_{\tau\to\mp\infty} H(\tau) = \pm 1$ and $\lim_{\tau\to\mp\infty} H(\tau) = \mp 1$ will be instanton and anti-instanton respectively. Moreover, in pure YM gauge theories like the one we consider here, the (anti)self-dual equation $F^a = \mp\tilde{F}^a$ always imply the Euler-Lagrange's equation of motion owing to the Bianchi identity. Thus we only need to solve this (anti)self-dual equation to obtain the solutions. And as we shall see shortly, (anti)instanton solutions emerge as explicit solutions to (anti)self-dual equation depending on the choice of gauge for the lapse function $N(\tau)$ associated with the time reparametrization invariance of the theory of background gravity. Note also that (anti)self-dual equation, $(dH/d\tau) = \mp(N/a)(H^2 - 1)$ exactly coincides with the first integral of the equation of motion with $E = 0$, $(dH/d\tilde{\tau}) = \mp[U(H)]^{1/2}$. In this Euclidean formulation, Euclidean time τ is just another spacelike coordinate and hence the instanton can be thought of as a soliton configuration (actually this is why 'tHooft dubbed the name "instantons" for the Euclidean solitons). Thus for later use, here we provide the expression for the energy of the instanton as the "soliton energy". Since the Euclidean action represents the energy of the system, the soliton energy is given by

$$
\begin{aligned}
\varepsilon_{soliton} &= I_{YM}[instanton] = \frac{r_0^2}{2}\int_{-\infty}^{\infty} d\tilde{\tau}[H'^2 + U(H)] \\
&= r_0^2\int_{-\infty}^{\infty} d\tau(\frac{N}{a})(H^2-1)^2 = -r_0^2\int_{-1}^{1} dH(H^2-1) \\
&= \frac{4}{3}r_0^2 = \frac{8\pi^2}{g_c^2}
\end{aligned}
\tag{19}
$$

where we used the first integral in eq.(18) of the Euclidean equation of motion with $E = 0$ and proper boundary conditions for instanton solutions. Note that this expression for the energy of the instanton displays a generic feature commonly shared by all soliton solutions [1], namely it is inversely proportional to the coupling constant of the theory, g_c, and hence the instanton is a non-perturbative object.

It is well-known that the vacuum in the pure YM theory in flat spacetime is infinitely degenerate and the degenerate vacua are classified by the topological structure (namely they fall into different homotopy classes). Thus one might be curious about the nature of changed vacuum structure in our case, i.e., now we have just two-fold degeneracy in the vacuum at $H = \pm 1$. It turns out that the two degenerate vacua $H = \pm 1$ in this pure YM theory formulated in the background of de Sitter spacetime are *not* associated with the non-trivial topology structure. Thus it seems worth comparing between the vacuum structure of YM theory in flat spacetime and that in the background of de Sitter spacetime. Firstly in the pure YM theory in flat spacetime, the vacuum corresponds to $F_{\mu\nu} = 0$, which, in terms of the gauge potential, is described by the "pure gauge"

$$
A_\mu = -\frac{i}{g_c}[\partial_\mu g(x)]g^{-1}(x) \tag{20}
$$

where $g(x) \in SU(2)$ denotes the SU(2) group-valued function. Now in Euclidean signature, the spacetime has the geometry and topology of R^4 and hence obviously its boundary at which the vacuum ($F_{\mu\nu} = 0$) occurs is S^3. Thus for the vacuum, the relevant base manifold for the SU(2) group-valued function $g(x)$ in the expression for the pure gauge above is S^3. Then the mapping $g(x)$ from the base manifold S^3 to the group manifold $SU(2) \sim S^3$ forms a non-trivial homotopy group $\Pi_3(SU(2)) = \Pi_3(S^3) = Z$. As a result, the YM theory vacuum in flat spacetime is infinitely degenerate and it consists of homotopically-inequivalent n-

vacua with n denoting the "winding number".

Secondly in the pure YM theory in the background of de Sitter spacetime which is the case at hand, however, the vacuum of the theory exhibits rather different nature. Namely, unlike the flat Euclidean spacetime, the background de Sitter spacetime represented by the spatially-closed ($k = +1$) FRW-metric has the topology of S^4. Therefore, the $k = +1$ FRW-metric itself and the YM gauge field defined on it are both taken to possess SO(4)-symmetry. And as we have seen, the two degenerate vacua occur for $H = \pm 1$ at $\tau = -\infty$ and for $H = \mp 1$ at $\tau = +\infty$. Namely for the vacuum, the relevant base manifold for the SU(2) group-valued function $g(x)$ now turns out to be, say, S^0 (0-sphere) consisting of two points $\{\tau = -\infty, \tau = +\infty\}$. Thus the mapping $g(x)$ from the base manifold S^0 to the group manifold $SU(2) \sim S^3$ forms trivial homotopy group $\Pi_0(S^3) = 0$. This suggests that the two degenerate vacua $H = \pm 1$ are not associated with the non-trivial homotopy structure. Rather, these two vacua can be thought of as an analogue of again two degenerate vacua existing in Wu-Yang magnetic monopole [8] in spherically- symmetric (i.e., SO(3)-symmetric) flat spacetime in which the monopole ansắtz for the gauge potential and its field strength are given in spherical-polar coordinates as

$$
\begin{aligned}
&A^a_t = A^a_r = 0,\\
&A^a_\theta = -\frac{1}{g_c}[1-u(r)]\hat{\phi}^a, \quad A^a_\phi = \frac{1}{g_c}[1-u(r)]\sin\theta\hat{\theta}^a
\end{aligned}
$$

and

$$
\begin{aligned}
F^a_{r\theta} &= \frac{u'(r)}{g_c}\hat{\phi}^a, \quad F^a_{r\phi} = -\frac{u'(r)}{g_c}\sin\theta\hat{\theta}^a,\\
F^a_{\theta\phi} &= \frac{[u^2(r)-1]}{g_c^2}\sin\theta\hat{r}^a.
\end{aligned}
$$

Obviously, the vacuum here, $F^a = 0$, amounts to two values $u(r) = \pm 1$. Again the vacuum of this system has two-fold degeneracy which is not of topological origin. Therefore, we can conclude that the two degenerate vacua $H(\tau) = \pm 1$ in our theory discussed above are of exactly the same kind. In the introduction we mentioned that we would give an account of the relationship between the instanton solution in our present work and that in the work of

Eguchi and Freund [2]. We will do it now. Eguchi and Freund [2] considered a Weyl-invariant theory with the action

$$S = \int d^4x\sqrt{g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}R\phi^2 - \lambda\phi^4 - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}]$$

which is indeed invariant under the Weyl rescaling

$$\begin{aligned} g_{\mu\nu} &= \Omega^2(x)\tilde{g}_{\mu\nu}, \quad x^\mu = \tilde{x}^\mu, \\ \phi &= \Omega^{-1}(x)\tilde{\phi}, \quad F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu} \end{aligned}$$

where ϕ is a scalar field. Then the classical field equations which result by extremizing this action with respect to $g_{\mu\nu}$, ϕ and A^a_μ admit a special (Euclidean) solution

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu}, \quad \phi = (\frac{2b^2}{\lambda})^{1/2}\frac{1}{(x^2+b^2)}, \\ A^a_\mu &= (A^a_\mu)_{BPST} \end{aligned}$$

where the subscript "BPST" indicates the flat space instanton solution. Now one can take advantage of the Weyl-invariance property of the theory. Namely, since the action has the Weyl-rescaling invariance given above, starting from this special solution, one can generates new solutions for every conformally-flat spacetime. For instance, de Sitter spacetime is conformally-flat and hence one might wish to construct the instanton solution in de Sitter spacetime. In order to make a transit to the de Sitter spacetime, one needs to force the scalar field ϕ in this theory to take the constant value $(3/4\pi G)^{1/2}$. This can be achieved by taking the conformal factor to be $\Omega(x) = (8\pi Gb^2/3\lambda)^{1/2}1/(x^2+b^2)$. Then the action above of this theory reduces to that of Einstein-Yang-Mills theory in the presence of the cosmological constant $\Lambda = (9\lambda/2\pi G)$. Now if we confine our interest only to the instanton solution in the YM sector, again the YM energy-momentum tensor vanishes identically and hence at the classical level this reduced system does represent a proper system to discuss the YM instanton solution in de Sitter spacetime and it (i.e., the instanton solution) turns out to remain the same as the flat space instanton solution. Now the questions is ; how do we compare this (unaffected) instanton solution in de Sitter spacetime with ours? As

a matter of fact, the Eguchi-Freund instanton solution above which was constructed via a conformal transformation like this is not quite the instanton solution in de Sitter spacetime in the rigorous sense. To see this, recall that the de Sitter spacetime is the space of (positive) constant curvature and hence is conformally-flat. And these characteristics imply that the de Sitter spacetime has the topology of S^4 in Euclidean signature. Now according to the method of Eguchi and Freund, one starts from the "seed" solution and perform a conformal transformation on it to obtain the solution in de Sitter spacetime. Since the metric solution in the seed solution is the flat metric with the topology of R^4, the conformal tranformation can *at most* turn it into the metric with the topology of $S^4 - \{p\}$ with $\{p\}$ denoting the "north pole", not the complete S^4. Thus the new solution is at most the instanton solution in the, say, "almost de Sitter" spacetime with the topology of $S^4 - \{p\}$ but not of complete S^4. This missing point $\{p\}$, then, indicates that the conformally-transformed metric $g_{\mu\nu} = \Omega^2(x)\delta_{\mu\nu}$ represents a non-compact spacetime with boundary corresponding to "points at infinity" of R^4 (this last point becomes manifest if one uses the stereographic projection). Namely, even after the conformal transformation, the background spacetime still has the boundary of topology of S^3 and thus the associated YM instanton solution maintains the non-trivial homotopy structure of its flat space counterpart. Our solution, on the other hand, is the YM instanton solution in *genuine* de Sitter spacetime with topology of complete S^4. As a result, as was pointed out earlier, the instanton solution in our system completely lacks the non-trivial homotopy structure. To conclude, the solution of Eguchi and Freund and that of ours are "two different" solutions which can not be related by any gauge transformation.

III. Classical Instanton Solutions

As is well-known, the classical solution to the (anti)self-dual equation $F_{\mu\nu} = \mp \tilde{F}_{\mu\nu}$ minimizes the Euclidean YM theory action [1],

$$I_{YM} = \int d^4x \sqrt{g} \frac{1}{2g_c^2} Tr(F_{\mu\nu}F_{\mu\nu}) \geq \pm \int d^4x \sqrt{g} \frac{1}{2g_c^2} Tr(F_{\mu\nu}\tilde{F}_{\mu\nu})$$

and thus makes dominant contribution to the vacuum-to-vacuum tunnelling amplitude. Thus here we attempt to solve (anti)self-dual equation to find instanton and anti-instanton

solutions. In terms of SO(4)-symmetric ansătz for the background metric and the YM gauge field on it, the (anti)self-dual equation takes the form given in eq.(12) which, as mentioned, coincides with the first integral of the Euclidean equation of motion with $E = 0$. Now we solve this (anti)self-dual equation with different gauge choices for the lapse function $N(\tau)$ associated with the time-reparametrization invariance of the background gravity.

(1) In the "conformal-time" gauge, $N(\tau) = a(\tau)$:

The (anti)self-dual equation in eq.(12) becomes, in this gauge

$$\frac{dH}{d\tau} = \mp(H^2 - 1) \tag{21}$$

which, upon integration, yields

$$H(\tau) = -\tanh\tau \tag{22}$$

as a solution of the self-dual equation and

$$H(\tau) = \tanh\tau \tag{23}$$

as a solution of the anti-self-dual equation.

Thus we have the instanton solutions

$$A^a = A^a_\mu dx^\mu = [1 \mp \tanh\tau]\sigma^a \tag{24}$$

(where $\mp$ indicates instanton and anti-instanton respectively) in the background de Sitter spacetime with metric

$$ds^2 = \frac{1}{\kappa^2}\frac{1}{\cosh^2\tau}[d\tau^2 + \sigma^a \otimes \sigma^a]. \tag{25}$$

(2) In the gauge, $N(\tau) = 1$:

The (anti)self-dual equation in eq.(12) becomes, in this gauge

$$\frac{dH}{d\tau} = \mp\frac{1}{a(\tau)}(H^2 - 1) \tag{26}$$

(with $a(\tau) = \frac{1}{\kappa}\cos(\kappa\tau)$) which, upon integration, yields

$$H(\tau) = -\sin(\kappa\tau) \tag{27}$$

as a solution of the self-dual equation and

$$H(\tau) = \sin(\kappa\tau) \tag{28}$$

as a solution of the anti-self-dual equation.

Note here that the cosmological constant Λ that determines the background spacetime as the classical de Sitter spacetime should be $\Lambda << M_p^2$ (where $M_p = G^{-1/2}$ denotes the Planck mass which sets the lower bound for the scale for quantum gravity) since the "background" spacetime is supposed to be a classical gravity with fixed geometry (and topology). This, in turn, implies that the period of the (anti)instanton solution above is infinite, i.e., $(period) = 2\pi/\kappa = 2\pi/\sqrt{8\pi\Lambda/3M_p^2} \to \infty$ and hence the shape of the (anti)instanton solution in this gauge $N(\tau) = 1$ is essentially the same as that of the (anti)instanton solution in the previous gauge $N(\tau) = a(\tau)$. Thus we have the instanton solutions

$$A^a = A^a_\mu dx^\mu = [1 \mp \sin(\kappa\tau)]\sigma^a \tag{29}$$

(where $\mp$ indicates instanton and anti-instanton respectively) in the background de Sitter spacetime with metric

$$ds^2 = [d\tau^2 + \frac{1}{\kappa^2}\cos^2(\kappa\tau)\sigma^a \otimes \sigma^a]. \tag{30}$$

Finally, note that regardless of the gauge choice for the lapse function, the (anti)instanton solutions are all $-1 \leq H(\tau) \leq +1$ and hence interpolate the two degenerate vacua $H = \pm 1$ as they should.

Upon constructing the classical (anti)instanton solutions explicitly, next we turn to the computation of their instanton number. Since we have constructed the single instanton and anti-instanton solutions above we anticipate that the assocated instanton number is $+1$ or -1 respectively. Recall that in the background of flat Euclidean spacetime, the instanton number is equal to the "Pontryagin index" or the "2nd Chern class" [1] given by

$$\nu[A] = \int_{R^4} d^4x \frac{-1}{16\pi^2} Tr[F_{\mu\nu}\tilde{F}_{\mu\nu}] \tag{31}$$

where Tr indicates the sum over repeated hidden group indices. Therefore, for the case of pure YM gauge theory in the background of curved spacetime like the present de Sitter case, one can similarly define the instanton number as the curved spacetime version of the Pontryagin index. Thus we shall compute this curved spacetime version of the Pontryagin index for our case. To do so, first note ;

$$\begin{aligned} F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} &= F^a_{AB}\tilde{F}^a_{AB} = 2F^a_{0b}\tilde{F}^a_{0b} + F^a_{bc}\tilde{F}^a_{bc} \\ &= [2(\frac{H'}{Na})(\frac{H^2-1}{a^2})\delta^a_b\delta^b_a + (\frac{H^2-1}{a^2})(\frac{H'}{Na})\epsilon^{abc}\epsilon_{abc}] \\ &= \frac{12}{Na^3}H'(H^2-1). \end{aligned} \tag{32}$$

Thus, the curved spacetime version of the Pontryagin index is

$$\begin{aligned} \nu[A] &= \int_{R\times S^3} d^4x\sqrt{g}\frac{-1}{32\pi^2}[F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}] \\ &= 2\pi^2\int_{-\infty}^{\infty} d\tau Na^3[\frac{-1}{32\pi^2}\{\frac{12}{Na^3}H'(H^2-1)\}] \\ &= \pm\frac{3}{2}\int_0^1 dH(H^2-1) \\ &= \pm 1 \end{aligned} \tag{33}$$

indicating that the instanton number is $+1$ for the single instanton solutions or -1 for the single anti-instanton solutions just as expected.

Now, as the final analysis of our classical instanton solution, we evaluate the instanton contribution to the vacuum-to-vacuum tunnelling amplitude. The "instanton action", namely the Euclidean action evaluated at the instanton is given by

$$\begin{aligned} I_{YM}(instanton) &= \int_{R\times S^3} d^4x\sqrt{g}[\frac{1}{4g_c^2}(F^a_{\mu\nu})^2] \mid_{instanton} \\ &= \frac{r_0^2}{2}\int_{-\infty}^{\infty} d\tau[(\frac{a}{N})H'^2 + (\frac{N}{a})(H^2-1)^2] \mid_{instanton} \\ &= r_0^2\int_{-\infty}^{\infty} d\tau(\frac{N}{a})(H^2-1)^2 \\ &= -2r_0^2\int_0^1 dH(H^2-1) = \frac{8\pi^2}{g_c^2} \end{aligned} \tag{34}$$

where we used the (anti)self-dual equation satisfied by (anti)instanton solution, $(dH/d\tau) = \mp(N/a)(H^2-1)$ and $r_0^2 = 6\pi^2/g_c^2$. This instanton action is essentially the same as the soliton energy we computed earlier in eq.(19). Consequently, the semiclassical approximation to the vacuum-to-vacuum transition amplitude is given by

$$\begin{aligned}(inter - vacua\ tunnelling\ amplitude) &\sim \exp\left[-I_{YM}(instanton)\right] \\ &= e^{-\frac{8\pi^2}{g_c^2}}.\end{aligned} \tag{35}$$

It is interesting to note that this instanton contribution to the inter-vacua tunnelling amplitude for the pure YM theory formulated in the background of de Sitter spacetime turns out to be the same as that for YM theory in the usual flat spacetime [1]. At this point, it seems appropriate to comment on the relationship between our present work and the work of Charap and Duff [2]. Charap and Duff [2] also considered Euclideanized Einstein-Yang-Mills theory (but in the absence of the cosmological constant) and looked for classical solutions to the (anti)self-dual equation in the YM sector. Again, since the YM field energy-momentum tensor vanishes for (anti)self-dual field strength in the Euclidean signature, the YM field does not disturb the spacetime geometry. Thus for the "background" geometry, they particularly took the Schwarzschild spacetime which has SO(3)-isometry and looked for instanton solutions which possess SO(3)-symmetry or O(4)-symmetry (having particularly the decomposition, $O(4) \approx SU(2) \times SU(2)$). It turned out that the resulting instanton solutions are characterized by the Pontryagin index $\nu[A] = \pm 1$ and have the instanton action $I_{YM}(instanton) = 8\pi^2/g_c^2$ which are the same as those for the instanton in flat space. Therefore the last point, namely that the curved spacetime version of instanton solutions carry the Pontryagin index of ± 1 and particularly that the inter-vacua tunnelling amplitude remains the same even when the gravity of certain type is turned on are precisely the same as the results of our study given in this section.

IV. Fermionic zero modes in the instanton background

In flat spacetime, one is usually interested in the dynamics of chiral fermions in the instanton background in order to explore phenomenon like chirality-changing fermion prop-

agation due to the background instanton configuration. Thus here we also consider the dynamics of chiral fermions in the background of our curved (i.e., de Sitter) spacetime version of instanton. Thus we begin by checking if there are fermionic zero modes in this curved spacetime instanton background.

The Dirac equations for massless SU(2) doublet fermion field in this curved spacetime instanton background are given by

$$\gamma^C e^\mu_C[\overrightarrow{\partial}_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB} - iA^a_\mu T^a]\Psi = 0, \tag{36}$$
$$\bar{\Psi}\gamma^C e^\mu_C[\overleftarrow{\partial}_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB} - iA^a_\mu T^a] = 0$$

where the covariant derivative is given by

$$\gamma^\mu \nabla_\mu = \gamma^\mu[\partial_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB} - iA^a_\mu T^a]$$

where $e^A_\mu(x)\Big(e^\mu_A(x)\Big)$ is the "vierbein" (and it's inverse) defined by $g_{\mu\nu}(x) = \delta_{AB}e^A_\mu(x)e^B_\nu(x)$ and $e^A_\mu e^\mu_B = \delta^A{}_B$, $e^\mu_A e^A_\nu = \delta^\mu{}_\nu$ and $e \equiv (det\ e^A_\mu)$. Thus the Greek indices μ, ν refer to coordinate basis while the Roman indices A,B = 0,1,2,3 refer to non-coordinate basis. In addition, $\gamma^\mu(x) = e^\mu_A(x)\gamma^A$ is the curved spacetime γ-matrices obeying $\{\gamma^\mu(x), \gamma^\nu(x)\} = -2g_{\mu\nu}(x)$ with γ^A being the usual flat spacetime γ-matrices. Next $(\partial_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB})$ is then the Lorentz covariant derivative with $\omega^A_{\mu B} = -e^\nu_B(\partial_\mu e^A_\nu - \Gamma^\lambda_{\mu\nu}e^A_\lambda)$ being the spin connection and $\sigma^{AB} = \frac{i}{2}[\gamma^A, \gamma^B]$ being the $SO(3,1)$ group generator in the spinor representation.

Now, in order eventually to confirm the Atiyah-Patodi-Singer index theorem [9] in our system under consideration, we begin with the brief review of the "mixed" anomaly and the associated index theorem in terms of Fujikawa's path integral formulation [10]. As usual, consider the chiral $U(1)_A$ transformation of the spinor field coupled to both the background gauge and gravitational field

$$\Psi(x) \to \Psi'(x) = e^{i\gamma_5\alpha(x)}\Psi(x), \tag{37}$$
$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}(x)e^{i\gamma_5\alpha(x)}.$$

Under this chiral transformation, the fermionic integration measure in the functional integral

$$Z = \int [d\Psi d\bar{\Psi}] e^{-S_F[\Psi,\bar{\Psi}]} \tag{38}$$

changes by the Jacobian determinant

$$[d\Psi d\bar{\Psi}] \to [d\Psi' d\bar{\Psi}'] = J(\alpha)[d\Psi d\bar{\Psi}] \tag{39}$$

with $J(\alpha) = \exp\left[-i \int d^{2n}x\alpha(x)A(x)\right]$ and as is well-known, this leads to the mixed (gauge + gravity) anomaly

$$\nabla_\mu < \bar{\Psi}\gamma^\mu\gamma_5\Psi >= i < \frac{1}{e}A(x) > \tag{40}$$

where $< ... >$ stands for expectation value in terms of the functional integral. And this "anomaly term" on the right hand side of eq.(40) is related to the Dirac index in the index theorem. Namely, the Atiyah-Patodi-Singer index theorem states that the analytical index defined by $(\text{index}\nabla)_{2n} = \text{dimker}(\nabla) - \text{dimker}(\nabla^\dagger)$ is just a topological invariant expressed in terms of an integral of an appropriate characteristic class over the $2n$-dim. manifold M^{2n}, i.e.,

$$\begin{aligned}(\text{index}\nabla)_{2n} &= \text{dimker}(\nabla) - \text{dimker}(\nabla^\dagger) \\ &= \int_{M^{2n}} \hat{A}(R)Ch(F)\end{aligned} \tag{41}$$

where $\hat{A}(R)$ and $Ch(F)$ denote "A-roof (or Dirac) genus" and the "total Chern character" respectively defined by

$$\begin{aligned}\hat{A}(R) &\equiv \prod_{i=1}^{n}\left[\frac{(x_i/2)}{\sinh(x_i/2)}\right], \qquad (x_i \equiv \frac{iR}{2\pi}), \\ Ch(F) &\equiv Tr\exp\left(\frac{iF}{2\pi}\right)\end{aligned} \tag{42}$$

with R and F being curvature and YM field strength 2-forms respectively. Generally in $2n$-dim., $\hat{A}(R)$ and $Ch(F)$ can be expanded in series of Pontryagin classes and Chern classes respectively and particularly in 4-dim., which is of our interest,

$$\begin{aligned}(\text{index}\nabla)_4 &= \frac{-1}{8\pi^2}\int_{M^4} Tr(F\wedge F) + \frac{1}{192\pi^2}\int_{M^4} Tr(R\wedge R) \\ &= \frac{-1}{16\pi^2}\int_{M^4} d^4x\sqrt{g}Tr(F_{\mu\nu}\tilde{F}_{\mu\nu}) - \frac{1}{8}\tau(M^4)\end{aligned} \tag{43}$$

where $\tau(M) \equiv \int_M (\frac{-1}{24\pi^2}) Tr(R \wedge R)$ denotes the "Hirzebruch signature" of the manifold M. For the case at hand, however, the background spacetime manifold M^4 is the Euclidean de Sitter space with the geometry and topology of that of S^4 whose Hirzebruch signature is $\tau(S^4) = 0$. Therefore, for massless fermions in the background of YM gauge field (particularly instantons) and in the background of Euclidean de Sitter space, the appropriate form of the Atiyah-Patodi-Singer index theorem reads

$$(\text{index}\nabla)_4 = \int_{S^4} d^4x \sqrt{g} \frac{-1}{16\pi^2} Tr(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \pm 1 \tag{44}$$

as has been evaluated earlier in eq.(33). Thus all we need to do is to confirm this relation by checking if there actually is at least one normalizable positive-chirality fermion zero mode or negative-chirality fermion zero mode. And to see this, the most straightforward way is to solve the Dirac equation for massless fermion field given earlier explicitly. In order to solve the Dirac equation, we need explicit expressions for the spin connection ω_μ^{AB} of de Sitter background spacetime (represented by $k = +1$ FRW-metric) and the YM gauge connection A_μ^a of the background instanton.

First, we can obtain the spin connection 1-forms, using the non-coordinate basis 1-forms given in eq.(4) and the Cartan's 1st structure equation (i.e., the torsion-free condition)

$$de^A + \omega_B^A \wedge e^B = 0 \tag{45}$$

along with the help of Maurer-Cartan structure equatin given in eq.(5). And they are

$$\omega_{\mu 0}^a = -\omega_{\mu a}^0 = \frac{1}{N}(\frac{a'}{a})e_\mu^a \quad , \quad \omega_{\mu b}^a = -\omega_{\mu a}^b = \frac{-1}{a}\epsilon^{abc}e_\mu^c. \tag{46}$$

Next, the YM gauge connection 1-form can be given in this non-coordinate basis as well. Namely, using $A^a = A_\mu^a dx^\mu = A_B^a e^B = A_b^a e^b = [(1+H)/a]e^a$ (since we chose the temporal gauge, $A_0 = 0$), we get

$$A_b^a = [\frac{1+H(\tau)}{a(\tau)}]\delta_b^a. \tag{47}$$

Now we are ready to solve the Dirac equation in eq.(37) using the concrete forms for the spin connection given in eq.(46) and for the YM gauge connection given in eq.(47). In addition, assuming that the fermion field depends only on the Euclidean time τ and setting

$$\Psi(\tau) = a^{-\frac{3}{2}}(\tau)\tilde{\Psi}(\tau), \tag{48}$$

the Dirac equation in eq.(37) reduces to

$$[\partial_\tau - \frac{N}{4a}\epsilon_{abc}\gamma^0\gamma^a\gamma^b\gamma^c - i\gamma^0\gamma^a T^a \frac{N}{a}(1+H)]\tilde{\Psi}(\tau) = 0. \tag{49}$$

Further using $\epsilon_{abc}\gamma^0\gamma^a\gamma^b\gamma^c = 3!\gamma_5$ (where $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is the Euclidean γ_5-matrix) and the original Dirac matrices $\gamma^0 = \beta$, $\gamma^a = \beta\alpha^a$ with matrices β, α^a satisfying $\{\alpha^a,\ \alpha^b\} = 2\delta^{ab}$, $\{\alpha^a,\ \beta\} = 0$ and $\beta^2 = I$, this Dirac equation can be rewritten as

$$[\partial_\tau - \gamma_5\frac{3N}{2a} - i\frac{1}{2}\alpha^a\tau^a\frac{N}{a}(1+H)]\tilde{\Psi}(\tau) = 0 \tag{50}$$

whose solution is readily given as

$$\tilde{\Psi}(\tau) = \exp\,[\frac{1}{2}\int^\tau d\tau'\frac{N}{a}\{3\chi + i(\alpha^a\tau^a)(1+H)\}]\ U \tag{51}$$

where $\chi = \pm 1$ depending on the constant basis spinor U which may have positive chirality, $\gamma_5 U_+ = U_+$ or negative chirality, $\gamma_5 U_- = -U_-$ respectively. Note that $[\gamma_5,\ \alpha^a] = 0$ since $\{\gamma_5,\ \beta\} = 0$, $\{\gamma_5,\ \beta\alpha^a\} = 0$. This implies that the chirality state of U-spinor exactly mirrors that of $\tilde{\Psi}(\tau)$ or $\Psi(\tau)$. Finally, the solution to the massless Dirac equation is found to be

$$\Psi(\tau) = \frac{1}{a^{\frac{3}{2}}(\tau)}\tilde{\Psi}(\tau).$$

And of course in this expression for the solution to the Dirac equation, the information of the background de Sitter space and the background instanton configuration is given by

$$a(\tau) = \frac{1}{\kappa}\cos(\kappa\tau), \qquad H(\tau) = \mp\sin\kappa\tau$$

for the gauge choice $N(\tau) = 1$ and

$$a(\tau) = \frac{1}{\kappa\cosh\tau}, \qquad H(\tau) = \mp\tanh\tau$$

for the gauge choice $N(\tau) = a(\tau)$ with the minus (plus) sign refering to instanton (anti-instanton) respectively. Since both $a(\tau)$ and $H(\tau)$ are bounded and oscillating functions of τ and the τ-integration in the exponent is finite due to the finite integration range, clearly

these massless solutions to the Dirac equation above are "normalizable" zero modes. Finally, since the Atiyah-Patodi-Singer index theorem states that

$$
\begin{aligned}
(\mathrm{index}\nabla)_4 &= \mathrm{dimker}(\nabla^\dagger\nabla) - \mathrm{dimker}(\nabla\nabla^\dagger) \\
&= n_+ - n_- = \pm 1
\end{aligned}
\tag{52}
$$

where we used eq.(44). Namely, the difference in the number of positive-chirality fermion zero modes (n_+) and that of negative-chirality fermion zero modes (n_-) cannot be arbitrary but is fixed by the Pontryagin index representing the instanton number. Thus for our case, for instanton with $\nu[A] = 1$, there should be, say, one positive-chirality fermion zero mode ($n_+ = 1$) with no negative-chirality zero mode ($n_- = 0$) while for anti-instanton with $\nu[A] = -1$, there should be one negative-chirality fermion zero mode ($n_- = 1$) with no positive-chirality zero mode ($n_+ = 0$). And in the above we have seen that this rule can indeed be obeyed since there is only one normalizable zero mode solution which could have either positive or negative chirality state.

V. Quantisation of instanton

(1) General description of soliton quantisation scheme

Before we carry out the explicit quantisation of the instanton in the YM theory formulated in the background of de Sitter spacetime represented by $k = +1$ FRW-metric, we provide a brief review of the conventional soliton quantisation scheme. Historically, the formalism for performing soliton quantisation has been developed in the original papers through a variety of techniques [5,6] and here in this work, we shall mainly refer to the formalism of Dashen et al. [5] which is generally known to be standard. In general, solitons can be associated with quantum extended-particle states. And certain properties of these quantum states like their energy, for instance, can be expanded in a semiclassical series. The leading terms in this series will be seen to be related to the corresponding classical soliton solutions. In this fashion, knowledge of the classical soliton solutions will yield some information about the quantum particle states, in a systematic semiclassical expansion. Moreover, this information will be non-perturbative in the non-linear couplings since, in most cases, the correspond-

ing classical solutions are themselves non-pertubative. Now, in order to demonstrate, in a general manner, the quantisation of static soliton to obtain extended, non-perturbative, quantum particle states, we consider a scalar field theory governed by the Lagrangian

$$L = \int d^3x [\frac{1}{2}(\frac{\partial \phi}{\partial t})^2 - \frac{1}{2}(\nabla \phi)^2 - U(\phi)] \tag{53}$$

The classical dynamics of this system is quite similar to particle mechanics. For instance, the Lagrangian has the familiar standard form $L = T[\phi] - V[\phi]$ with the kinetic and potential energy being given by

$$T[\phi] = \int d^3x \frac{1}{2}(\frac{\partial \phi}{\partial t})^2, \quad V[\phi] = \int d^3x [\frac{1}{2}(\nabla \phi)^2 + U(\phi)]. \tag{54}$$

Next, extremizing this Lagrangian yields the following Euler-Lagrange's equation of motion

$$\frac{\partial^2 \phi(t, \vec{x})}{\partial t^2} = -\frac{\delta V[\phi]}{\delta \phi(t, \vec{x})} \tag{55}$$

As a particle mechanical analogue, this classical field equation is similar to Newton's equation of motion with the field $\phi(t, \vec{x})$ playing the role of the "coordinates". First note that static solutions $\phi(t, \vec{x}) = \phi(\vec{x})$ satisfying

$$\frac{\delta V[\phi]}{\delta \phi(\vec{x})} = -\frac{\delta L}{\delta \phi(\vec{x})} = 0 \tag{56}$$

are automatically the extremum points for both the Lagrangian and the potential energy $V[\phi]$ in field space. In particular, stable static solutions such as the vacuum or soliton solutions are "minima" of $V[\phi]$ just as in particle mechanics. Let $\phi(\vec{x}) = \phi_0(\vec{x})$ be one such minimum, then we can make a functional Taylor expansion of $V[\phi]$ about ϕ_0 at which $V[\phi]$ gets minimized ;

$$V[\phi] = V[\phi_0] + \int d^3x \frac{1}{2!}\{\eta(\vec{x})[-\nabla^2 + (\frac{d^2U}{d\phi^2}) |_{\phi_0(\vec{x})}]\eta(\vec{x}) + ...\} \tag{57}$$

where $\eta(\vec{x}) \equiv \phi(\vec{x}) - \phi_0(\vec{x})$, and integration by parts has been used and 'dots' represent cubic and higher terms. These higher order terms would be small and thus can be neglected or treated in perturbation provided the magnitude of the fluctuations $\eta(\vec{x})$ is small and/or the

third and higher derivatives of $V[\phi]$ at ϕ_0 are small. Thus to lowest order in this approximation expansion, eigenvalues and eigenfunctions of the operator $[-\nabla^2 + (d^2U/d\phi^2)\mid_{\phi_0}]$ will be given by the following differential equation

$$[-\nabla^2 + (\frac{d^2U}{d\phi^2})\mid_{\phi_0(\vec{x})}]\eta_i(\vec{x}) = \omega_i^2\eta_i(\vec{x}) \tag{58}$$

where $\eta_i(\vec{x})$ are the orthonormal "normal modes" of fluctuations around $\phi_0(\vec{x})$. Then next, following Creutz [11], introduce

$$\eta(t,\vec{x}) \equiv \phi(t,\vec{x}) - \phi_0(\vec{x}) \equiv \sum_i C_i(t)\eta_i(\vec{x}). \tag{59}$$

Then using the orthogonality, $\int d^3x\eta_i(\vec{x})\eta_j(\vec{x}) = \delta_{ij}$, the Lagrangian of this system becomes

$$L = \frac{1}{2}\sum_i[\dot{C}_i(t)]^2 - \left(V[\phi_0] + \frac{1}{2}\sum_i[C_i(t)]^2\omega_i^2\right) + ... \tag{60}$$

where $\dot{C}_i \equiv dC_i/dt$ and dots stand for contributions from higher terms. Evidently, this reduced Lagrangian represents that of a set of harmonic oscillators, one for each normal mode, apart from a constant term $V[\phi_0]$. For the sake of definiteness, if we take the usual ϕ^4-theory as an example, $U(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$. Then applying the above general formulation with the choice of the stable static solution $\phi_0(t,\vec{x}) = 0$ leads to $\eta i(\vec{x}) = \frac{1}{\sqrt{L^3}}\exp{(i\vec{k}_i\cdot\vec{x})}$ and $\omega_i^2 = (\vec{k}_i^2 + m^2)$ with $k_iL = 2\pi N_i$ $(L\to\infty)$ in the box-normalization which amounts to a quantisation condition. Correspondingly, in quantum theory one can construct a set of approximate harmonic oscillator states with energies given by

$$E_{\{n_i\}} = V[\phi_0] + \hbar\sum_i(n_i + \frac{1}{2})[\vec{k}_i^2 + m^2]^{1/2} \tag{61}$$

where n_i is the excitation number of the i-th normal mode. It relates, approximately, the energies of certain quantum levels to the classical solution $\phi_0(\vec{x})$. The second term involves ω_i, which are the stability frequencies of $\phi_0(\vec{x})$. This completes the short review of the standard soliton quantisation scheme that we shall employ in the case of our interest.

(2) Quantisation of the instanton

Now, returning to our problem, consider the Euclidean action of our system, i.e., the pure YM theory in de Sitter background spacetime given earlier,

$$I_{YM} \equiv V[H] = \frac{r_0^2}{2} \int d\tilde{\tau}[(\frac{dH}{d\tilde{\tau}})^2 + (H^2-1)^2]. \tag{62}$$

Since the "Euclidean time", $\tilde{\tau}$ is just another "spacelike" coordinate, this Euclidean action may be viewed as the potential energy $V[H]$ or the Hamiltonian H_{YM}, in the above general formalism, of a system of a scalar field $H(\tilde{\tau})$ with the "double-well" potential $U(H) = \frac{1}{2}(H^2-1)^2$. Namely, our system can be viewed as a kind of scalar H^4-theory in a stable static soliton sector. Therefore one can apply the standard soliton quantization scheme described above to the quantisation of vacuum and instantons of our theory. To be more specific, we would like to explore the energy spectrums of excitations around the vacuum and the instanton configurations.

A. The vacuum and its excitations

We first begin with the excitations around the classical vacuum (i.e., one of the two degenerate vacua), $H_1(\tau) = 1$. The Euclidean action actually represents the energy of this system and it can be expanded around the vacuum $H_1 = 1$ as

$$\begin{aligned} I_{YM}[H] = I_{YM}[H_1] + r_0^2\{&\int d\tilde{\tau}\frac{1}{2}\tilde{H}[-\frac{\partial^2}{\partial\tilde{\tau}^2}+4]\tilde{H} \\ &+ \int d\tilde{\tau} 2\tilde{H}^3 + \int d\tilde{\tau}\frac{1}{2}\tilde{H}^4 + O(\tilde{H}^5)\} \end{aligned} \tag{63}$$

where $\tilde{H} \equiv (H - H_1) = H - 1$ and $I_{YM}[H_1] = 0$. Note that since r_0^2 is an overall factor commonly multiplied to all terms in the Euclidean action (or the Hamiltonian), we will henceforth work with the rescaled Euclidean action $I'_{YM} = I_{YM}/r_0^2$ and then restore this overall factor r_0^2 at the end of the computation of the energies. Now, if we restrict our interest to excitations arising from sufficiently small deviations from the classical vacuum, i.e., $\tilde{H} = (H-1) << 1$, then terms higher than the cubic term can be ignored. Then in the lowest- order quadratic term, the second functional derivative of $I'_{YM}[H]$ at $H = 1$ is the operator $[-\frac{\partial^2}{\partial\tilde{\tau}^2}+4]$, whose eigenvalues are (k_n^2+4) with eigenfunctions $e^{ik_n\tilde{\tau}}$. Then the allowed values of k_n are obtained, in box-normalization, by

$$k_n L = 2\pi n \tag{64}$$

where $n \in Z$ and L, the length of the box, will ultimately tend to infinity with the replacement,

$$\sum_{k_n} \Longrightarrow L \int \frac{dk}{(2\pi)}. \tag{65}$$

Now, we can construct a tower of approximate harmonic oscillator states around the vacuum $H_1 = 1$, the lowest of which has the energy, restoring the overall factor r_0^2,

$$E_{vac} \cong 0 + \frac{1}{2}\hbar r_0^2 \sum_n [k_n^2 + 2^2]^{1/2} \tag{66}$$

where the zero represents the classical vacuum energy $I_{YM}[H_1 = 1]$. This is the quantum state of the vacuum of the system. Next, higher excitations will have energies

$$E_{vac} \cong \hbar r_0^2 \sum_n (N_n + \frac{1}{2})[k_n^2 + 2^2]^{1/2}. \tag{67}$$

These correspond to the familiar quanta of the theory, where N_n of them have momentum $\hbar k_n$. We will call this set of states built around the vacuum $H_1 = 1$, the "vacuum sector". Since this procedure essentially quantizes the shifted field, $\tilde{H} = (H - H_1) = (H - 1)$, as in standard perturbation methods, we can borrow the familiar result to lowest order that

$$< 0|\tilde{H}(\tau)|0 >= 0 \qquad \text{or} \qquad < 0|H(\tau)|0 >= H_1 = 1 \tag{68}$$

where $|0>$ denotes the vacum state.

B. The quantum instanton and its excitations

Next, we turn to the excitations around the instanton configuration. For the sake of definiteness, we choose to work with the instanton solution resulting from the gauge fixing $N(\tau) = a(\tau)$

$$H_c(\tau) = \tanh \tau$$

with energy (i.e., Euclidean action evaluated at this instanton solution)

$$I_{YM}[H_c] = \frac{8\pi^2}{g_c^2}.$$

Clearly, this instanton solution is an extremum point of the Euclidean action $I_{YM}[H]$. Thus again, the Euclidean action, namely the energy of the system can be expanded around this instanton solution, i.e., the extremum point $H_c(\tau)$ as

$$\begin{aligned} I_{YM}[H] = I_{YM}[H_c] + r_0^2\{\int d\tilde{\tau}\frac{1}{2}\tilde{H}[-\frac{\partial^2}{\partial\tilde{\tau}^2} - 2 + 6H_c^2]\tilde{H} \\ + \int d\tilde{\tau}[2H_c\tilde{H}^3 + \frac{1}{2}\tilde{H}^4] + O(\tilde{H}^5)\} \end{aligned} \tag{69}$$

where now $\tilde{H} \equiv (H - H_c)$ and $d\tilde{\tau} = d\tau(N/a) = d\tau$ since this instanton solution corresponds to the gauge choice $N(\tau) = a(\tau)$. Henceforth, again, we shall work with the rescaled Euclidean action $I'_{YM} = I_{YM}/r_0^2$. In the lowest-order quadratic term, the eigenvalues of the second functional derivative of $I_{YM}[H]$ at H_c are given by the equation

$$\begin{aligned} & [-\frac{\partial^2}{\partial\tau^2} - 2 + 6H_c^2]\tilde{H}_n(\tau) \\ & = [-\frac{\partial^2}{\partial\tau^2} - 2 + 6\tanh^2\tau]\tilde{H}_n(\tau) = \omega_n^2\tilde{H}_n(\tau). \end{aligned} \tag{70}$$

Dividing this equation through by 2, it becomes a Schrődinger-type equation

$$[-\frac{1}{2}\frac{\partial^2}{\partial\tau^2} + (3\tanh^2\tau - 1)]\tilde{H}_n(\tau) = \frac{\omega_n^2}{2}\tilde{H}_n(\tau). \tag{71}$$

Fortunately, the eigenfunctions and eigenvalues of this Schrődinger-type equation are exactly known [12]. It has two discrete levels followed by a continuum. The discrete levels are ;

$$\begin{aligned} \omega_0^2 = 0 \qquad \text{with} \qquad \tilde{H}_0(\tau) = \frac{1}{\cosh^2\tau} \\ \omega_1^2 = 3 \qquad \text{with} \qquad \tilde{H}_1(\tau) = \frac{\sinh}{\cosh^2\tau} \end{aligned} \tag{72}$$

This is followed by a continuum of levels which we shall label by q rather than by $n \geq 2$. These are ;

$$\omega_q^2 = [q^2 + 2^2] \qquad \text{with} \qquad \tilde{H}_q(\tau) = e^{iq\tau}[3\tanh^2\tau - 1 - q^2 - 3iq\tanh\tau]. \tag{73}$$

Here, the allowed values of q, like the allowed values of k_n in the vacuum case, are fixed by periodic boundary conditions in a box of length L, with $L \to \infty$. It is noteworthy that the

quantum fluctuation or excitation around the classical instanton solution $\tilde{H}_q(\tau)$ above has an asymptotic behavior

$$\tilde{H}_q(\tau) \longrightarrow \exp\left[i(q\tau \pm \frac{1}{2}\delta(q))\right] \qquad \text{as} \quad \tau \to \pm\infty$$

where $\delta(q) = -2\arctan[3q/(2-q^2)]$ is just the phase shift of the scattering states of the associated Schrődinger problem above. This is precisely the quantum fluctuation around the classical vacuum,

$$\tilde{H}_n(\tau) = e^{ik_n\tau} \qquad \text{with} \qquad \omega_n^2 = [k_n^2 + 2^2]$$

we obtained earlier modulo phase shift just as expected since $\tau \to \pm\infty$ is the vacuum limit. Now, as before, the allowed values of q are determined by thr periodic boundary condition in box-normalization

$$q_n L + \delta(q_n) = 2\pi n, \qquad n \in Z. \tag{74}$$

In the $L \to \infty$ limit, these allowed values merge into a continuum with the replacement

$$\sum_{q_n} \Longrightarrow \int_{-\infty}^{\infty} \frac{dq}{(2\pi)}[L + \frac{\partial}{\partial q}\delta(q)]. \tag{75}$$

Now we are ready to write down (or construct) the energy spectrum of quantized instanton as a sum of the energy of the classical instanton and the energy levels of the small quantum fluctuations (excitations) around that classical instanton. Notice that in the expansion of the Euclidean action, i.e., the energy of the system around the instanton, namely its extremum point $H_c(\tau)$, if we restrict our interest to excitations arising from sufficiently small deviation from the classical instanton, i.e., $\tilde{H} \equiv (H - H_c) << 1$, terms higher than the cubic term can be neglected and we are left with the minimum energy (i.e., energy of the classical instanton) and quadratic term in $\tilde{H}(\tau)$. Then this lowest-order quadratic term can be identified with the one representing energy levels of a set of approximate harmonic oscillator states spread in field space around $H_c(\tau)$ with the neglected higher-order terms representing all anharmonic terms. Therefore, in this approximation of quantum fluctuations around the

classical instanton by a set of harmonic oscillator states in field space, the energy spectrum of quantized instanton is given by (restoring the overall factor r_0^2)

$$E_{\{N_n\}} \cong I_{YM}[H_c] + \hbar r_0^2 \sum_{n=0}^{\infty} (N_n + \frac{1}{2})\omega_n \tag{76}$$
$$= \frac{4}{3}r_0^2 + (N_1 + \frac{1}{2})\hbar r_0^2 \sqrt{3} + \hbar r_0^2 \sum_{q_n} (N_{q_n} + \frac{1}{2})[q_n^2 + 2^2]^{1/2}$$

where, as we did in the quantized vacuum case, we explicitly retain $\hbar$ for a few steps since we wish to bring out the semiclassical nature of our theory. There is, however, a point to which one should be cautious ; while this analysis used in the treatment of small quantum fluctuations around the classical instanton is essentially valid for all the $n \geq 1$ modes, it does not hold for the $n = 0$ mode, because $\omega_0 = 0$. Namely, unlike the $n \geq 1$ modes which are genuine vibrational modes for small fluctuations, the $n = 0$ mode is not vibrational at all. The 'spring constant' ω_0 vanishes. Correspondingly, the quantum wave function along the $n = 0$ mode will not be confined near a given classical solution, but will tend to spread. Now we turn to the interpretation of the energy spectrum of the tower of quantized instanton states given above ;

(i) The lowest-energy state,

$$E_0 \equiv E_{\{N_n=0\}} = \frac{4}{3}r_0^2 + \frac{1}{2}\hbar r_0^2 \sqrt{3} + \frac{1}{2}\hbar r_0^2 \sum_{q_n} [q_n^2 + 2^2]^{1/2} \tag{77}$$

may be interpreted as the "lowest energy state of the quantum instanton". Note that although this state has lowest energy in the 'instanton sector', obviously it is not the absolute ground state or vacuum of this theory. As we already discussed earlier, the vacuum of this theory has been identified as the lowest energy state in the 'vacuum sector',

$$E_{vac} \cong \frac{1}{2}\hbar r_0^2 \sum_{n} [k_n^2 + 2^2]^{1/2}. \tag{78}$$

(ii) The next higher energy level,

$$E_1 \equiv E_{\{N_1=1;N_{q_n}=0\}} = E_0 + \hbar r_0^2 \sqrt{3} \tag{79}$$

may be interpreted as a discrete excited state of the quantum instanton. And higher excitations of this mode (i.e., $N_1 > 1$) give higher excited states of the quantum instanton.

(iii) The remaining higher energy states obtained by exciting the $n \geq 2$ modes (i.e., the $N_q \neq 0$ states) can be thought of as scattering states "quanta in the vacuum sector" in the presence of the quantum instanton.

This is our interpretation of the family of quantum instanton states constructed around the classical instanton solution. Next, since we have constructed the quantum instanton states (i.e., small quantum fluctuations around the classical instanton solution) and the associated energy spectrum, naturally the next question we would like to ask is ; what would the effects of this quantum instanton on the vacuum-to-vacuum tunnelling amplitude be ? Namely, we would like to explore the lowest quantum correction to the Euclidean action and hence to the vacuum-to-vacuum tunnelling amplitude arising from the quantisation of the instanton. Since the saddle point (instanton) approximation to the inter-vacua tunnelling amplitude is given by

$$\Gamma \sim e^{-I_{YM}[instanton]}, \tag{80}$$

one can naively expect that the tunnelling amplitude involving the quantum correction coming from the contribution from the quantum instanton at its lowest energy state (restoring the overall factor r_0^2)

$$E_0 = \frac{4}{3}r_0^2 + \frac{1}{2}\hbar r_0^2\sqrt{3} + \frac{1}{2}\hbar r_0^2 \sum_{q_n}[q_n^2 + 2^2]^{1/2}$$

would be given by

$$\Gamma \sim e^{-E_0}.$$

Unfortunately, however, this naive prescription fails since the expression for the lowest energy above is formally divergent. The infinite series over $\sum_{q_n}$ in the last term of E_0 above becomes, in the continuum limit,

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi}[L + \frac{\partial}{\partial q}\delta(q)][q^2 + 2^2]^{1/2}$$

i.e., a quadratically-divergent integral. This in itself, however, need not worry us since the lowest energy of the quantum vacuum state is also quadratically divergent again when the infinite series over $\sum_{k_n}$ is taken over to the continuum limit, viz.,

$$E_{vac} \cong \frac{1}{2}\hbar r_0^2 \sum_n [k_n^2 + 2^2]^{1/2} \tag{81}$$
$$= \frac{1}{2}\hbar r_0^2 L \int_{-\infty}^{\infty} \frac{dk}{(2\pi)}[k^2 + 2^2]^{1/2} \rightarrow \infty.$$

After all, what matters physically is the difference in energy between any given state and the vacuum state. And this difference is obtained by subtracting E_{vac} from E_0 (From now on we will discuss the regularization and the renormalization of the lowest energy of the quantized instanton state. Since the overall factor r_0^2 is irrelevant in the regularization procedure, we will henceforth work with the rescaled energy $E' = E/r_0^2$ associated with the rescaled Euclidean action $I'_{YM} = I_{YM}/r_0^2$. And at the end of the computations, we will restore the overall factor r_0^2 in the final expression for the renormalized value of the lowest energy of the quantized instanton.),

$$E'_0 - E'_{vac} = \frac{4}{3} + \frac{1}{2}\hbar\sqrt{3} + \frac{1}{2}\hbar \sum_n [(q_n^2 + 4)^{1/2} - (k_n^2 + 4)^{1/2}]. \tag{82}$$

Since both terms in the bracket are divergent, we must subtract them carefully so as not to lose finite pieces. Let us start with a finite box with size L. As usual, the periodic boundary condition in the box-normalization determines the allowed values of k_n and q_n as

$$2\pi n = k_n L = q_n L + \delta(q_n). \tag{83}$$

Thus, the term in bracket in $(E'_0 - E'_{vac})$ becomes

$$\{[(k_n - \frac{\delta_n}{L})^2 + 4]^{1/2} - [k_n^2 + 4]^{1/2}\} = -(\frac{k_n \delta_n}{L})(k_n^2 + 4)^{-1/2} + O(\frac{1}{L^2}) \tag{84}$$

where $\delta_n \equiv \delta(q_n)$. Now going to the $L \rightarrow \infty$ limit and using the replacement

$$\sum_{k_n} \Longrightarrow L \int \frac{dk}{(2\pi)},$$

we have

$$E'_0 - E'_{vac} = \frac{4}{3} + \frac{1}{2}\hbar\sqrt{3} - \frac{\hbar}{4\pi}\int_{-\infty}^{\infty} dk \frac{k\delta(k)}{\sqrt{k^2 + 4}} \tag{85}$$

where using $\delta(q) = -2\arctan[3q/(2 - q^2)]$ and $q_n = (k_n - \delta_n/L)$,

$$\delta(k) = -2\arctan[\frac{3k}{(2-k^2)}] + O(\frac{1}{L}). \tag{86}$$

Then next upon integrating by parts, we get

$$\begin{aligned} E_0' - E_{vac}' &= \frac{4}{3} + \frac{1}{2}\hbar\sqrt{3} - \frac{\hbar}{4\pi}[\delta(k)\sqrt{k^2+4}]_{-\infty}^{\infty} + \frac{\hbar}{4\pi}\int_{-\infty}^{\infty} dk\sqrt{k^2+4}\frac{d}{dk}[\delta(k)] \\ &= \frac{4}{3} + \frac{1}{2}\hbar\sqrt{3} - 3\hbar\pi - \frac{3\hbar}{2\pi}\int_{-\infty}^{\infty} dk\frac{(k^2+2)}{\sqrt{k^2+4}(k^2+1)} \end{aligned} \tag{87}$$

where we used the phase shift in eq.(86). Now, although the quadratic divergence in E_0' has been removed by subtracting out E_{vac}', $(E_0' - E_{vac}')$ still has a logarithmic divergence in the last term involving integral. In fact, this divergence at this stage of the calculation need not concern us. We actually should expect it to be there and it can be removed by "normal ordering" the Hamiltonian. The occurrence of ultraviolet divergences in quantum field theory due to the short-distance behavior of products of field operators is well-known in standard perturbation theory. And typically, these divergences are removed by adding suitable "counter terms" to the Hamiltonian. Now for our theory, the Euclidean action which is equivalent to the Hamiltonian is

$$I_{YM}' = H_{YM}' = \frac{1}{2}\int d\tilde{\tau}[(\frac{dH}{d\tilde{\tau}})^2 + H^4 - 2H^2 + 1]. \tag{88}$$

In the quantised theory, operators like $H^2(\tau)$, $H^4(\tau)$ etc. are formally divergent and ill-defined and thus so is the Hamiltonian. As a consequence, energy levels calculated naively from this Hamiltonian will be divergent as well. And this is the very reason behind the divergence in $(E_0' - E_{vac}')$. Now the removal of such divergences can be accomplished by replacing the Hamiltonian by its normal ordered form $: H' :$. In our semiclassical formulation, however, it would be difficult to work directly with the normal-ordered form. Instead, using the results of Wick's theorem, the normal-ordered form can be written as the original non-ordered form plus some counter terms,

$$\begin{aligned} : H^4 : &= H^4 - AH^2 - B, \\ : H^2 : &= H^2 - C \end{aligned} \tag{89}$$

where A,B and C are constants which diverge in perturbation theory. Therefore, the normal-ordered Hamiltonian may be written as (setting $A \equiv \partial m^2$ and $D \equiv B - 2C$)

$$
\begin{aligned}
:H'_{YM}: \; &= H'_{YM} - \int_{-\infty}^{\infty} d\tilde{\tau}[\partial m^2 H^2 + D] \\
&\equiv H'_{YM} + \Delta E'
\end{aligned}
\tag{90}
$$

where the constants ∂m^2 and D may be evaluated in perturbation theory by standard methods. In particular, ∂m^2 is the renormalization constant in the mass renormalization and to 1-loop order, it is given by

$$
\partial m^2 = \frac{12\hbar}{16\pi}\int_{-\Lambda}^{\Lambda}\frac{dk}{\sqrt{k^2 - m^2}} = \frac{3\hbar}{4\pi}\int_{-\Lambda}^{\Lambda}\frac{dk}{\sqrt{k^2+2}} \tag{91}
$$

where the numerical factor 12 comes from the combinatorial factor of associating each scalar field operator in $H^4(\tau)$ each with line in the Feynman diagram and Λ is the momentum cut-off. Also we used the fact that in our scalar $H(\tau)$-field system represented by the Euclidean action or the Hamiltonian given in eq.(90), the mass squared corresponds to $m^2 = -2$ and finally the factor $\hbar$ represents the 1-loop correction. Next, we will not evaluate the other renormalization constant D since $(E'_0 - E'_{vac})$ involves the difference between two energy levels where the effect of D will cancel out. Now we are ready to demonstrate that the counter term

$$
\Delta E' = -\int_{-\infty}^{\infty} d\tilde{\tau}[\partial m^2 H^2 + D]
$$

indeed removes the logarithmic divergence in $(E'_0 - E'_{vac})$. Since the replacement of the Hamiltonian (or the Euclidean action) H'_{YM} by its normal ordered form $:H'_{YM}:$ amounts to adding the counter term $\Delta E'$ above, in order to renormalize the lowest energy of the quantum instanton state $(E'_0 - E'_{vac})$, we have to add it by the counter terms

$$
\begin{aligned}
(\Delta E'_0 - \Delta E'_{vac}) &= -\int_{-\infty}^{\infty} d\tilde{\tau}[\partial m^2 H_c^2(\tilde{\tau}) + D] + \int_{-\infty}^{\infty} d\tilde{\tau}[\partial m^2 H_1^2(\tilde{\tau}) + D] \\
&= (\partial m^2)\int_{-\infty}^{\infty} d\tau[1 - \tanh^2\tau] \\
&= 2(\partial m^2) = \frac{3\hbar}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{dk}{\sqrt{k^2+2}}.
\end{aligned}
\tag{92}
$$

Actually, this is the leading contribution of the counter terms to $(E_0' - E_{vac}')$ since we inserted classical instanton $H_c(\tau)$ and classical vacuum $H_1(\tau)$ into the "quantum" field $H(\tau)$ appearing in the counter term $\Delta E'$ above. Therefore finally, the finite, renormalized lowest energy of the quantum instanton state is given by [5], upon restoring the overall factor r_0^2,

$$\begin{aligned}
E_0^{ren} &\equiv (E_0 - E_{vac}) + (\Delta E_0 - \Delta E_{vac}) \\
&= \frac{4}{3}r_0^2 + \hbar r_0^2(\frac{\sqrt{3}}{2} - \frac{3}{\pi}) - \frac{3\hbar}{2\pi}r_0^2 \int_{-\infty}^{\infty} dk[\frac{(k^2+2)}{\sqrt{k^2+4}(k^2+1)} - \frac{1}{\sqrt{k^2+2}}] \\
&= \frac{4}{3}r_0^2 + \hbar r_0^2(\frac{\sqrt{3}}{6} - \frac{3}{\pi}) + O(\hbar^3). \qquad (93)
\end{aligned}$$

Note here that both terms in the integrand behave as $1/k$ as $k \to \infty$ so that the logarithmic divergences cancel out. To conclude, the "renormalized" (i.e., free of infinities of all sorts) lowest energy of the quantized instanton state is given by

$$E_0^{ren} = r_0^2[\frac{4}{3} - \hbar(\frac{3}{\pi} - \frac{\sqrt{3}}{6})] = \frac{8\pi^2}{g_c^2}[1 - \hbar\frac{3}{4}(\frac{3}{\pi} - \frac{\sqrt{3}}{6})]. \qquad (94)$$

We now make a few remarks on the renormalized value of the lowest energy of the quantum instanton state given above. The first term $4r_0^2/3 = 8\pi^2/g_c^2$ is the Euclidean action evaluated at the classical instanton solution i.e., the energy of the classical instanton. The next term represents the leading correction coming from quantum fluctuations. Thus appropriately, the first term is of order $\hbar^0$ and the second term is of order $\hbar^1$. Secondly, in the weak-coupling limit, $g_c << 1$, thanks mainly to the energy of the classical instanton, this lowest energy of the quantised instanton is much larger than the lowest energy of the quanta in the vacuum sector which is of order of $\hbar r_0^2 = \hbar 6\pi^2/g_c^2$. In addition, it seems worth mentioning that what we have done so far to get eq.(94) is the renormalization of the energy E_0 of the quantized instanton, *not* the usual renormalization of the gauge coupling constant g_c of the theory. Finally we return to our major concern, namely the computation of the lowest quantum correction to the Euclidean action and hence to the vacuum-to-vacuum tunnelling amplitude arising from the quantisation of the instanton. Note that the renormalized lowest energy of the quantized instanton state E_0^{ren} given above is *lower* than the energy (Euclidean action) of the classical instanton, i.e.,

$$E_0^{ren} = \frac{8\pi^2}{g_c^2}[1 - \hbar\frac{3}{4}(\frac{3}{\pi} - \frac{\sqrt{3}}{6})] < I_{YM}[instanton] = \frac{8\pi^2}{g_c^2} \tag{95}$$

which then implies that

$$\Gamma_{QI} \sim e^{-E_0^{ren}} > \Gamma_{CI} \sim e^{-I_{YM}[instanton]} \tag{96}$$

namely the inter-vacua tunnelling amplitude gets *enhanced* upon quantizing the instanton. Obviously, this is an expected result since the tunnelling between degenerate vacua is really a quantum phenomenon in nature. Notice that our system, i.e., the pure YM theory in the background of de Sitter spacetime represented by $k = +1$ FRW-metric exhibits, albeit in a much simpler structure, almost all of the features of the YM theory in flat spacetime as being unchanged including particularly the same vacuum-to-vacuum tunnelling amplitude in the instanton approximation. Thus we believe that the estimate of the lowest quantum correction to the Euclidean instanton action and hence to the vacuum-to-vacuum tunnelling amplitude arising from the quantisation of the instanton would remain the same even if we ask the same question in the pure YM theory in flat spacetime although the actual computation of the corresponding quantity (i.e., E_0^{ren}) would be even more formidable there!

VI. Discussions

We now summarize the results of the present work. In this work, we examined, in detail, the instantons and their quantisation in pure YM theory formulated in the background of de Sitter spacetime represented by spatially-closed ($k = +1$) FRW-metric. The SO(4)-symmetry of the $k = +1$ FRW-metric having the topology of S^4 and hence that of the dynamical YM field put on it, effectively reduced the system to that of an one-dimensional self-interacting scalar field theory with double-well potential. Since the vacuum structure of this reduced system has just two-fold degeneracy and thus is relatively simple, the associated instanton physics could be analyzed in a quantitative manner. Classical instanton configurations have been obtained as explicit solutions to the (anti)self-dual equation which implies the Euclidean YM equation of motion. In addition, the Pontryagin index representing the instanton number and the inter-vacua tunnelling amplitude associated with these instanton

solutions have been evaluated. The single instanton and the single anti-instanton are found to possess Pontryagin index $+1$ and -1 respectively, as expected. In particular, it is remarkable that the semiclassical approximation (involving only the instanton contribution) to the vacuum-to-vacuum tunnelling amplitude for our YM theory formulated in de Sitter background spacetime turned out to be the same as that for YM theory in the usual flat spacetime. Atiyah-Patodi-Singer index theorem was also checked in our system by demonstrating explicitly that there is only one normalizable fermion zero mode in this de Sitter spacetime instanton background which has either positive or negative chirality state. Lastly, we attempted the quantisation of our instanton solution using the fact that the action or the Hamiltonian of our reduced one-dimensional system takes on precisely the same structure as that of one-dimensional scalar field theory which admits kink soliton solutions. Therefore, following the kink quantisation programme originally proposed by Dashen, Hasslacher and Neveu, we performed the quantisation of the vacuum and the instanton of our theory. Of particular interest was the estimation of the lowest quantum correction to the Euclidean action and hence to the vacuum-to-vacuum tunnelling amplitude arising from the quantisation of the instanton. It turned out that the renormalized lowest energy of the quantized instanton state is lower than the energy of the classical instanton. As a consequence, the inter-vacua tunnelling amplitude gets enhanced upon quantizing the instanton.

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